Multipole Expansion of Radiation Fields

Harmonic fields

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})e^{-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}(\vec{x})e^{-i\omega t}$$

In the radiation zone, $\vec{J} = \rho = 0$.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = ik\vec{B} \qquad k = \frac{\omega}{c}$$
$$\vec{\nabla}\times\vec{B} = -ik\vec{E}$$

Eliminate $\vec{E}$

$$\vec{\nabla}\times\frac{i}{k}(\vec{\nabla}\times\vec{B}) = ik\vec{B}$$

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{B}(\vec{x}) = 0}$$

homogeneous Helmholtz equations

Eliminate $\vec{B}$

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{E}(\vec{x}) = 0}$$

<u>Theorem:</u> Let $\vec{F}$ be any vector field satisfying $\vec{\nabla}\cdot\vec{F}=0$. Then there exist scalar functions ψ, χ such that

$$\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$$

where

ψ, χ = <u>Debye potentials</u>

$$\vec{L} = -i\vec{x}\times\vec{\nabla}$$

ψ, χ are unique up to

arbitrary radial function

$$\psi(\vec{x}) \to \psi(\vec{x}) + f(r) \qquad r \equiv |\vec{x}|$$

$$\chi(\vec{x}) \to \chi(\vec{x}) + g(r)$$

Note that

$$\vec{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1) Y_{\ell m}(\theta,\phi)$$

<u>Operator identities</u>

$$\vec{L}\cdot(\vec{\nabla}\times\vec{L}) = 0 \qquad \vec{\nabla}\cdot\vec{L} = 0$$

$$\vec{x}\cdot\vec{L} = 0 \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{L}) = 0$$

$$\vec{x}\cdot(\vec{\nabla}\times\vec{L}) = i\vec{L}^2$$

Hence, using $\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$

$$\vec{x}\cdot\vec{F} = i\vec{L}^2\chi$$

$$\vec{L}\cdot\vec{F} = \vec{L}^2\psi$$

Solve for χ, ψ by expanding in spherical harmonics

Note that if $\vec{L}^2 f(\vec{x}) = 0$, then

$f(\vec{x}) = f(r)$ is a radial function

Hence $\chi \to \chi + g(r)$, $\psi \to \psi + f(r)$ do not modify $\vec{x}\cdot\vec{F}$ and $\vec{L}\cdot\vec{F}$.

<u>Note</u>: $\vec{L}\cdot\vec{F} = -i\vec{x}\cdot(\vec{\nabla}\times\vec{F})$

Thus, χ, ψ are determined by $\vec{x}\cdot\vec{F}$ and $\vec{x}\cdot(\vec{\nabla}\times\vec{F})$.

To complete the proof of the theorem, one must show that if $\vec{\nabla}\cdot\vec{F} = 0$ and $\vec{x}\cdot\vec{F} = \vec{x}\cdot(\vec{\nabla}\times\vec{F}) = 0$ then $\vec{F} = 0$. This can be seen by noting that since $\vec{x}\cdot\vec{F} = 0$, the lines of "force" $\vec{F}$ are tangential on a sphere of radius R. $\vec{\nabla}\cdot\vec{F} = 0$ implies that there are no sources, and lines of force do not cross. Such a field cannot satisfy $\vec{x}\cdot(\vec{\nabla}\times\vec{F}) = 0$, unless $\vec{F} = 0$.

Second theorem

If $\vec{\nabla}\cdot\vec{F}=0$, $(\vec{\nabla}^2+k^2)\vec{F}=0$

then
$$\vec{F}=\vec{L}\psi+(\vec{\nabla}\times\vec{L})\chi$$

such that
$$(\vec{\nabla}^2+k^2)\psi=0$$
$$(\vec{\nabla}^2+k^2)\chi=0$$

Proof: Recall that

$$\psi(\vec{x})\longrightarrow\psi(\vec{x})+f(r) \qquad r\equiv|\vec{x}|$$
$$\chi(\vec{x})\longrightarrow\chi(\vec{x})+g(r)$$

because $\vec{L}=-i\vec{x}\times\vec{\nabla}$ is pure angular differential operator

$$\vec{L}f(r)=0.$$

For example,
$$\vec{x}\cdot\vec{F}=i\vec{L}^2\chi(\vec{x})$$

$$\vec{\nabla}^2(\vec{x}\cdot\vec{F}) = \vec{x}\cdot(\vec{\nabla}^2\vec{F}) + 2\vec{\nabla}\cdot\vec{F}$$

$$= \vec{x}\cdot(\vec{\nabla}^2\vec{F})$$

$$= -k^2\vec{x}\cdot\vec{F}$$

$$(\vec{\nabla}^2+k^2)\,\vec{x}\cdot\vec{F} = 0$$

$$(\vec{\nabla}^2+k^2)\,\vec{L}^2\chi(\vec{x}) = \vec{L}^2(\vec{\nabla}^2+k^2)\chi(\vec{x}) = 0$$

Hence, $(\vec{\nabla}^2+k^2)\,\chi(\vec{x}) = h(r)$

Finally, let $\chi(\vec{x}) \to \chi(\vec{x}) + g(r)$ where

$$(\vec{\nabla}^2+k^2)\,g(r) = -h(r)$$

$$\Longrightarrow (\vec{\nabla}^2+k^2)\left[\chi(\vec{x})+g(r)\right] = h(r)-h(r)$$

$$= 0$$

<u>Note:</u>

$$g(r) = -(\vec{\nabla}^2+k^2)^{-1}h(r)$$

$$= \frac{1}{4\pi}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,h(r')\,d^3x'$$

Recall:

$$(\vec{\nabla}^2+k^2)\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = -4\pi\delta^3(\vec{x}-\vec{x}')$$

Similarly, show that

$$\vec{\nabla}^2(\vec{x}\cdot(\vec{\nabla}\times\vec{F})) = -k^2(\vec{x}\cdot(\vec{\nabla}\times\vec{F}))$$

$$(\vec{\nabla}^2+k^2)\,\vec{x}\cdot(\vec{\nabla}\times\vec{F}) = 0$$

Then use $\vec{x}\cdot(\vec{\nabla}\times\vec{F}) = \vec{x}\cdot\vec{\nabla}\times[\vec{L}\psi+(\vec{\nabla}\times\vec{L})\chi]$

$$= \underbrace{\vec{x}\cdot(\vec{\nabla}\times\vec{L})}_{i\vec{L}^2}\psi + \underbrace{\vec{x}\cdot\vec{\nabla}\times(\vec{\nabla}\times\vec{L})\chi}_{0}$$

$$\vec{x}\cdot(\vec{\nabla}\times\vec{F}) = i\vec{L}^2\psi$$

$$\Longrightarrow \quad \vec{L}^2(\vec{\nabla}^2+k^2)\psi = 0$$

$(\vec{\nabla}^2+k^2)\psi$ is a radial function.

<u>Conclusion</u>:

$$\vec{x}\cdot\vec{F} = i\vec{L}^2\chi$$

$$\vec{x}\cdot(\vec{\nabla}\times\vec{F}) = i\vec{L}^2\psi$$

and $(\vec{\nabla}^2+k^2)\psi = (\vec{\nabla}^2+k^2)\chi = 0$

where $\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$

Remarks on the differential operator $\vec{L} = -i\vec{x}\times\vec{\nabla}$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi}$$

$$L_\pm = L_x \pm i L_y = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2}$$

$$\vec{L} = -i(\vec{x}\times\vec{\nabla}) = i\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

<u>exercise:</u> show that $[\vec{L}^2, L_i]$ $\quad i = x, y, z$

$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$

The multipole expansion of $\vec{B}, \vec{E}$

$$\boxed{\vec{B}(\vec{x}) = \vec{L}\psi^E(\vec{x}) - \frac{i}{k}(\vec{\nabla}\times\vec{L})\psi^M(\vec{x})}$$

$$\vec{E}(\vec{x}) = \frac{i}{k}\vec{\nabla}\times\vec{B}$$

conventional

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{L}) = -\vec{L}\,\vec{\nabla}^2 = -\vec{\nabla}^2\vec{L}$$

$$\boxed{\vec{E}(\vec{x}) = \vec{L}\,\psi^M(\vec{x}) + \frac{i}{k}(\vec{\nabla}\times\vec{L})\,\psi^E(\vec{x})}$$

where $(\vec{\nabla}^2 + k^2)\,\psi^E(\vec{x}) = 0$

$$(\vec{\nabla}^2 + k^2)\,\psi^M(\vec{x}) = 0$$

In spherical coordinates

$$\vec{\nabla}^2 + k^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2} + k^2$$

$$\vec{L}^2\,Y_{\ell m}(\theta,\phi) = \ell(\ell+1)\,Y_{\ell m}(\theta,\phi)$$

To solve $(\vec{\nabla}^2 + k^2)\,\psi(\vec{x}) = 0$

$$\psi(\vec{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(r)\,Y_{\ell m}(\theta,\phi)$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) f_{\ell m}(r) = 0$$

<u>remark:</u> The sum over ℓ actually starts at $\ell = 1$.

Solution:

$$f_{\ell m}(r) = a_{\ell m}\, j_\ell(kr) + b_{\ell m}\, n_\ell(kr)$$

$$j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$$

$$n_\ell(x) \equiv \sqrt{\frac{\pi}{2x}}\, N_{\ell+\frac{1}{2}}(x)$$

$$h_\ell^{(1)}(x) \equiv \sqrt{\frac{\pi}{2x}}\left[J_{\ell+\frac{1}{2}}(x) + i\, N_{\ell+\frac{1}{2}}(x)\right]$$

$$h_\ell^{(2)}(x) \equiv \sqrt{\frac{\pi}{2x}}\left[J_{\ell+\frac{1}{2}}(x) - i\, N_{\ell+\frac{1}{2}}(x)\right]$$

Explicitly

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\sin x}{x}\right)$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \left(\frac{\cos x}{x}\right)$$

For example:

$$j_0(x) = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

etc.

Large argument limit $\quad x \to \infty$

$$j_\ell(x) \longrightarrow \frac{1}{x} \sin\left(x - \frac{\ell\pi}{2}\right)$$

$$n_\ell(x) \longrightarrow -\frac{1}{x} \cos\left(x - \frac{\ell\pi}{2}\right)$$

$$h_\ell^{(1)}(x) \longrightarrow (-i)^{\ell+1} \frac{e^{ix}}{x}$$

$x = kr$

outgoing spherical waves $(r \to \infty)$

$$\sim \frac{e^{ikr}}{r}$$

Conclusion

$$\psi^E(\vec{x}) = \sum_{\ell m} \psi^E_{\ell m} h_\ell^{(1)}(kr) Y_{\ell m}(\theta, \phi)$$

$$\psi^M(\vec{x}) = \sum_{\ell m} \psi^M_{\ell m} h_\ell^{(1)}(kr) Y_{\ell m}(\theta, \phi)$$

sum starts at $\ell = 1$

$$\vec{B}(\vec{x}) = \sum_{\ell m} \psi^E_{\ell m} h_\ell^{(1)}(kr) \vec{L} Y_{\ell m}(\Omega)$$

$$+ \sum_{\ell m} \psi^M_{\ell m} \left(\frac{-i}{k}\right) \vec{\nabla} \times \vec{L}\, h_\ell^{(1)}(kr)\, Y_{\ell m}(\Omega)$$

$$\vec{E}(\vec{x}) = \sum_{\ell m} \psi^E_{\ell m} \frac{i}{k} \vec{\nabla} \times \vec{L}\, h_\ell^{(1)}(kr) Y_{\ell m}(\Omega)$$

$$+ \sum_{\ell m} \psi^M_{\ell m} h_\ell^{(1)}(kr) \vec{L} Y_{\ell m}(\Omega)$$

In the radiation zone, $r \to \infty$

$$h_\ell^{(1)}(kr) = (-i)^{\ell+1} \frac{e^{ikr}}{kr}$$

<u>Electric (ℓ, m) multipole</u> start at $\ell = 1$

$$\vec{B}^{(E)}_{\ell m} = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L} Y_{\ell m}(\theta, \phi)$$

$$\vec{E}^{(E)}_{\ell m} = \frac{i}{k} \vec{\nabla} \times B^{(E)}_{\ell m}$$

Magnetic (ℓ, m) multipole

start at $\ell = 1$

$$\vec{E}^{(M)}_{\ell m} = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L} Y_{\ell m}(\theta, \phi)$$

$$\vec{B}^{(M)}_{\ell m} = -\frac{i}{k} \vec{\nabla} \times \vec{E}^{(M)}_{\ell m}$$

Vector spherical harmonics

$$\vec{X}_{\ell m}(\theta, \phi) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m}(\theta, \phi)$$

$\ell = 1, 2, \ldots$

$m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

$$\int d\Omega \, \vec{X}^*_{\ell' m'}(\theta, \phi) \cdot \vec{X}_{\ell m}(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}$$

Note $\vec{\nabla} \cdot \vec{X}_{\ell m}(\theta, \phi) = 0$ since $\vec{\nabla} \cdot \vec{L} = 0$

second vector spherical harmonic $i \vec{\nabla} \times \vec{L} Y_{\ell m}(\theta, \phi)$

Last time, we identified the electric multipoles

(ℓ, m) $\quad \ell = 1, 2, 3, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

with magnetic and electric fields given by

$$\vec{B}(\vec{x}) = \psi^E_{\ell m} \vec{B}^{(E)}_{\ell m}(\vec{x}) \qquad \vec{B}(\vec{x},t) = \vec{B}(\vec{x}) e^{-i\omega t}$$

$$\vec{E}(\vec{x}) = \psi^E_{\ell m} \vec{E}^{(E)}_{\ell m}(\vec{x}) \qquad \vec{E}(\vec{x},t) = \vec{E}(\vec{x}) e^{-i\omega t}$$

where
$$\vec{B}^{(E)}_{\ell m}(\vec{x}) = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L} Y_{\ell m}(\Omega)$$

$$\vec{E}^{(E)}_{\ell m}(\vec{x}) = \frac{i}{k} \vec{\nabla} \times \vec{B}^{(E)}_{\ell m}$$

in the radiation zone $(kr \gg 1)$.

note:
$\hat{n} \cdot \vec{B} = 0$ using
$\hat{n} \cdot \vec{L} = 0$

The vector spherical harmonics are defined by

$$\vec{X}_{\ell m}(\Omega) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m}(\Omega)$$

$$\vec{L} = -i\vec{x} \times \vec{\nabla} = -ir\hat{n} \times \vec{\nabla}$$

To make contact with Jackson's notation

$$a_E(\ell, m) \equiv \sqrt{\ell(\ell+1)}\, \psi^E_{\ell m}$$

Likewise, the magnetic multipoles fields are

$$\vec{E}(\vec{x}) = \psi^M_{\ell m} \vec{E}^{(M)}_{\ell m}(\vec{x})$$

$$\vec{B}(\vec{x}) = \psi^M_{\ell m} \vec{B}^{(M)}(\vec{x})$$

where
$$\vec{E}^{(M)}_{\ell m}(\vec{x}) = (-i)^{\ell+1} \frac{e^{ikr}}{kr} \vec{L} Y_{\ell m}(\Omega)$$

$$\vec{B}^{(m)}_{\ell m}(\vec{x}) = -\frac{i}{k}\vec{\nabla}\times\vec{E}^{(m)}_{\ell m}$$

in the radiation zone ($kr \gg 1$).

Making contact with Jackson's notation,

$$a_m(\ell,m) \equiv \sqrt{\ell(\ell+1)}\,\psi^m_{\ell m}$$

Radiated power

$$P = \oint \vec{S}\cdot d\vec{a}$$

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\,\mathrm{Re}(\vec{E}\times\vec{B}^*)\cdot\hat{n}\, r^2$$

(i) electric multipole radiation (ℓ,m)

Need

$$\frac{i}{k}\vec{\nabla}\times\left[\frac{e^{ikr}}{r}\vec{L}Y_{\ell m}(\Omega)\right]$$

$$= \frac{i}{k}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\vec{L}Y_{\ell m}(\Omega) + \frac{e^{ikr}}{kr}\, i\vec{\nabla}\times\vec{L}Y_{\ell m}(\Omega)$$

useful operator identity

$$i\vec{\nabla}\times\vec{L} = \vec{x}\,\vec{\nabla}^2 - \vec{\nabla}\left(1 + r\frac{\partial}{\partial r}\right)$$

$$\vec{\nabla}\left(\frac{e^{ikr}}{r}\right) = \hat{n}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) = ik\frac{e^{ikr}}{r}\hat{n} + o\left(\frac{1}{r^2}\right)$$

$$\frac{\partial}{\partial r} Y_{\ell m}(\Omega) = 0$$

Thus,

$$\frac{i}{k}\vec{\nabla}\times\left[\frac{e^{ikr}}{r}\vec{L}\,Y_{\ell m}(\Omega)\right]$$

$$= -\frac{e^{ikr}}{kr}\left[k\hat{n}\times\vec{L}\,Y_{\ell m}(\Omega) - \overbrace{(\vec{x}\vec{\nabla}^2 - \vec{\nabla})Y_{\ell m}(\Omega)}^{o\left(\frac{1}{r^2}\right)}\right] + o\left(\frac{1}{r^2}\right)$$

$$= -\frac{e^{ikr}}{r}\hat{n}\times\vec{L}\,Y_{\ell m}(\Omega) + o\left(\frac{1}{r^2}\right)$$

$$\left[\text{e.g. } \vec{\nabla} = \hat{n}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$

That is,

$$\vec{E}^{E}_{\ell m} = -\hat{n}\times\vec{B}^{E}_{\ell m} + o\left(\frac{1}{r^2}\right)$$

$$\vec{E}\times\vec{B}^* = -(\hat{n}\times\vec{B})\times\vec{B}^* = \hat{n}|\vec{B}|^2 - \vec{B}(\hat{n}\cdot\vec{B}^*)$$

$$= \hat{n}|\vec{B}|^2$$

$$\vec{S}\cdot\hat{n}\cdot r^2 d\Omega = \frac{c}{8\pi k^2}\,|\psi^E_{\ell m}|^2\,|\vec{L}\,Y_{\ell m}(\Omega)|^2\,d\Omega$$

$$\boxed{\frac{dP_{E\ell m}}{d\Omega} = \frac{c}{8\pi k^2}\,|\psi^E_{\ell m}|^2\,\ell(\ell+1)\,|\vec{X}_{\ell m}(\Omega)|^2}$$

Integrating over angles

$$\int d\Omega\,\vec{X}^*_{\ell' m'}(\Omega)\cdot\vec{X}_{\ell m}(\Omega) = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$\Longrightarrow \int d\Omega\,|\vec{X}_{\ell m}(\Omega)|^2 = 1$$

$$\boxed{P_{E\ell m} = \frac{c\,\ell(\ell+1)}{8\pi k^2}\,|\psi^E_{\ell m}|^2}$$

(ii) magnetic multipole radiation

$$\vec{B}\to\vec{E}\,,\quad \vec{E}\to-\vec{B}$$

$$\boxed{\frac{dP_{M\ell m}}{d\Omega} = \frac{c}{8\pi k^2}\,|\psi^M_{\ell m}|^2\,\ell(\ell+1)\,|\vec{X}_{\ell m}(\Omega)|^2}$$