Multipole Expansion of Radiation Folds

Harmonic fields

$$
\begin{aligned}
& \vec{E}(\vec{x}, t)=\vec{E}(\vec{x}) e^{-i \omega t} \\
& \vec{B}(\vec{x}, t)=\vec{B}(\vec{x}) e^{-i \omega t}
\end{aligned}
$$

In the radiation zone, $\vec{J}=\rho=0$.
Maxwell's equations:

$$
\begin{aligned}
& \vec{\nabla} \cdot \vec{E}=\vec{D} \cdot \vec{B}=0 \\
& \vec{\nabla} \times \vec{E}=i k \vec{B} \quad k=\frac{\omega}{c} \\
& \vec{D} \times \vec{B}=-i k \vec{E}
\end{aligned}
$$

Eliminate $\vec{E}$

$$
\frac{\vec{\nabla} \times \frac{i}{k}(\vec{\nabla} \times \vec{B})=i k \vec{B}}{\left(\vec{\nabla}^{2}+k^{2}\right) \vec{B}(\vec{x})=0}
$$

homogeneous Helmholtz equations

$$
\left(\vec{\nabla}^{2}+k^{2}\right) \vec{E}(\vec{x})=0
$$

Theorem: Let $\vec{F}$ be any vector feed satisfying $\vec{\nabla} \cdot \vec{F}=0$. Then there exist scalar functions $\psi, \mathcal{X}$ such that

$$
\vec{F}=\vec{L} \psi+(\vec{\nabla} \times \vec{L}) X
$$

where
$\psi, X=$ Debye potentials

$$
\vec{L}=-i \vec{x} \times \vec{\nabla}
$$

$4, x$ are unique up to
arbitrary radial function

$$
\begin{array}{lr}
\psi(\vec{x}) \rightarrow \psi(\vec{x})+f(r) & r \equiv|\vec{x}| \\
X(\vec{x}) \rightarrow \chi(\vec{x})+g(r) &
\end{array}
$$

Note that

$$
\vec{L} y_{e m}(\theta, \phi)=l(l+1) Y_{e m}(\theta, \phi)
$$

Operator identities

$$
\begin{array}{ll}
\vec{L} \cdot(\vec{\nabla} \times \vec{L})=0 & \vec{\nabla} \cdot \vec{L}=0 \\
\vec{x} \cdot \vec{L}=0 & \vec{\nabla} \cdot(\vec{\nabla} \times \vec{L})=0 \\
\vec{x} \cdot(\vec{\nabla} \times \vec{L})=i \vec{L}^{2} &
\end{array}
$$

Hence, using $\vec{F}=\vec{L} \psi+(\vec{D} \times \vec{L}) \chi$

$$
\begin{aligned}
& \vec{X} \cdot \vec{F}=e^{2} X \\
& \vec{L} \cdot \vec{F}=\vec{L}^{2} \psi
\end{aligned}
$$

Solve for $x, 4$ by expanding in sphencal harmonics
Note that if $\vec{L}^{2} f(\vec{x})=0$, then

$$
f(\vec{x})=f(r) \text { is a radial function }
$$

Hence $X \rightarrow X+g(r), \psi \rightarrow \psi+f(r)$ do not modify $\vec{x} \cdot \vec{F}$ and $\vec{L} \cdot \vec{F}$.

Note: $\vec{L} \cdot \vec{F}=-e \vec{x} \cdot(\vec{\nabla} \times \vec{F})$
Thus, $X, \psi$ are determined by $\vec{x} \cdot \vec{F}$ and $\vec{x} \cdot(\vec{D} \times \vec{F})$.

To complete the proof of the theorem, one must show that if $\vec{\nabla} \cdot \vec{F}=0$ and $\vec{x} \cdot \vec{F}=\vec{x} \cdot(\vec{D} \times \vec{F})=0$ then $\vec{F}=0$. This can be seen by noting that since $\vec{x} \cdot \vec{F}=0$, the lives of "force" $\vec{F}$ are tangential on a sphere of radius $R$. $\vec{\nabla} \cdot \vec{F}=0$ implies that there are no sources, and lines of force do not cross. Such a field cannot satisfy $\vec{x} \cdot(\vec{D} \times \vec{F})=0$, unless $\vec{F}=0$.

Second theorem
If $\vec{\nabla} \cdot \vec{F}=0, \quad\left(\vec{\nabla}^{2}+k^{2}\right) \vec{F}=0$
Then

$$
\vec{F}=\vec{L} \psi+(\vec{D} \times \vec{L}) X
$$

such that $\quad\left(\vec{\nabla}^{2}+k^{2}\right) \psi=0$

$$
\left(\vec{D}^{2}+k^{2}\right) \chi=0
$$

Proof: Recall that

$$
\begin{array}{ll}
\psi(\vec{x}) \rightarrow \psi(\vec{x})+f(r) & r \equiv|\vec{x}| \\
x(\vec{x}) \rightarrow x(\vec{x})+g(r) &
\end{array}
$$

because $\vec{L}=-i \vec{x} \times \vec{D}$ is pure angular differential operator

$$
\vec{L} f(r)=0
$$

For example,

$$
\vec{x} \cdot \vec{F}=1 \vec{L}^{2} \chi(\vec{x})
$$

$$
\begin{aligned}
& \vec{\nabla}^{2}(\vec{x} \cdot \vec{F})=\vec{x} \cdot\left(\vec{\nabla}^{2} \vec{F}\right)+2 \vec{\nabla} \cdot \vec{F} \\
&=\vec{x} \cdot\left(\vec{\nabla}^{2} \vec{F}\right) \\
&=-k^{2} \vec{x} \cdot \vec{F} \\
&\left(\vec{\nabla}^{2}+k^{2}\right) \vec{x} \cdot \vec{F}=0 \\
&\left(\vec{\nabla}^{2}+k^{2}\right) \vec{L}^{2} X(\vec{x})=\vec{L}^{2}\left(\vec{\nabla}^{2}+k^{2}\right) X(\vec{x})=0
\end{aligned}
$$

Hence, $\left(\vec{D}^{2}+k^{2}\right) x(\vec{x})=h(r)$
Finally, let $X(\vec{x}) \rightarrow X(\vec{x})+g(r)$ where

$$
\begin{aligned}
&\left(\vec{\nabla}^{2}+k^{2}\right) g(r)=-h(r) \\
& \Rightarrow\left(\vec{\nabla}^{2}+k^{2}\right)[X(\vec{x})+g(r)]=h(r)-h(r) \\
&=0
\end{aligned}
$$

Note:

$$
\begin{aligned}
g(r) & =-\left(\vec{D}^{2}+k^{2}\right)^{-1} h(r) \\
& =\frac{1}{4 \pi} \int \frac{e^{i k\left|\vec{x}-\vec{x}^{\prime}\right|}}{\left|\vec{x}-\vec{x}^{\prime}\right|} h\left(r^{\prime}\right) d^{3} x^{\prime}
\end{aligned}
$$

Recall:

$$
\left(\vec{\nabla}^{2}+k^{2}\right) \frac{e^{i k\left|\vec{x}-\vec{x}^{\prime}\right|}}{\left|\vec{x}-\vec{x}^{\prime}\right|}=-4 \pi \delta^{3}\left(\vec{x}-\vec{x}^{\prime}\right)
$$

Similarly, show that

$$
\begin{aligned}
\vec{\nabla}^{2}(\vec{x} \cdot(\vec{\nabla} \times \vec{F})) & =-k^{2}(\vec{x} \cdot(\vec{D} \times \vec{F})) \\
\left(\vec{\nabla}^{2}+k^{2}\right) \vec{x} \cdot(\vec{\nabla} \times \vec{F}) & =0
\end{aligned}
$$

Then use

$$
\begin{aligned}
\vec{x} \cdot(\vec{D} \times \vec{F}) & =\vec{x} \cdot \vec{\nabla} \times[\vec{L} \psi+(\vec{D} \times \vec{L}) x] \\
& =\underbrace{\vec{x} \cdot(\vec{\nabla} \times \vec{L})}_{i \vec{L}^{2}} \psi+\underbrace{\vec{x} \cdot \vec{D} \times(\vec{D} \times \vec{L}) x}_{0}
\end{aligned}
$$

$$
\vec{x} \cdot(\vec{\nabla} \times \vec{F})=e \overrightarrow{L^{2}} \psi
$$

$$
\Rightarrow \quad \vec{L}^{2}\left(\vec{\nabla}^{2}+k^{2}\right) \psi=0
$$

$\left(\vec{\nabla}^{2}+k^{2}\right) \psi$ is a radial function.

Conclusion:

$$
\begin{gathered}
\vec{x} \cdot \vec{F}=1 \vec{L}^{2} X \\
\vec{x} \cdot(\vec{\nabla} \times \vec{F})=1 \vec{L}^{2} \psi
\end{gathered}
$$

and $\left(\vec{\nabla}^{2}+k^{2}\right) \psi=\left(\vec{\nabla}^{2}+k^{2}\right) \chi=0$
where $\vec{F}=\vec{L} \psi+(\vec{\nabla} \times \vec{L}) X$

Remarks on the differential operator $\vec{L}=-2 \vec{x} \times \vec{D}$

$$
\begin{aligned}
& L_{z}=-i\left(x \frac{\partial}{\partial y}-y \frac{\partial}{\partial x}\right)=-i \frac{\partial}{\partial \phi} \\
& L_{ \pm}=L_{x} \pm i L_{y}=e^{ \pm i \phi}\left( \pm \frac{\partial}{\partial \theta}+i \cot \theta \frac{\partial}{\partial \phi}\right) \\
& \vec{L}^{2}=L_{x}^{2}+L_{y}^{2}+L_{z}^{2}=-\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\right)-\frac{1}{\sin ^{2} \theta} \frac{\partial^{2}}{\partial \phi^{2}} \\
& \vec{\nabla}^{2}=\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r} \frac{\partial}{\partial r}-\frac{\vec{L}^{2}}{r^{2}} \\
& \vec{L}=-i(\vec{x} \times \vec{\nabla})=i\left(\hat{\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}-\hat{\phi} \frac{\partial}{\partial \theta}\right)
\end{aligned}
$$

exercise: Show that $\left[\vec{L}^{2}, L_{i}\right] \quad i=x, y, z$

$$
\left[L_{i}, L_{j}\right]=i \varepsilon_{i j k} L_{k}
$$

The multipole expansion of $\vec{B}, \vec{E}$

$$
\begin{aligned}
& \vec{B}(\vec{x})=\vec{L} \psi^{E}(\vec{x})-\frac{i}{k}(\vec{\nabla} \times \vec{L}) \psi^{m}(\vec{x}) \\
& \vec{E}(\vec{x})=\frac{i}{k} \vec{\nabla} \times \vec{B} \quad \hat{\tau} \text { conventional }
\end{aligned}
$$

$$
\begin{gathered}
\vec{\nabla} \times(\vec{\nabla} \times \vec{L})=-\vec{L} \vec{\nabla}^{2}=-\vec{\nabla}^{2} \vec{L} \\
\vec{E}(\vec{x})=\vec{L} \psi^{m}(\vec{x})+\frac{i}{k}(\vec{\nabla} \times \vec{L}) \psi^{E}(\vec{x})
\end{gathered}
$$

where $\left(\vec{\nabla}^{2}+k^{2}\right) \psi^{E}(\vec{x})=0$

$$
\left(\vec{\nabla}^{2}+k^{2}\right) \psi^{m}(\vec{x})=0
$$

In sphencal coordinates

$$
\begin{aligned}
& \vec{\nabla}^{2}+k^{2}=\frac{\partial^{2}}{\partial r^{\sigma}}+\frac{2}{r} \frac{\partial}{\partial r}-\frac{\vec{L}^{2}}{r^{2}}+k^{2} \\
& \vec{L}^{2} Y_{e m}(\theta, \phi)=l(\ell+1) Y_{e m}(\theta, \phi)
\end{aligned}
$$

To solve $\left(\vec{D}^{2}+k^{2}\right) \psi(\vec{x})=0$

$$
\begin{gathered}
\psi(\vec{x})=\sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{e m}(r) Y_{e m}(\theta, \phi) \\
\left(\frac{d^{2}}{d r^{2}}+\frac{2}{r} \frac{d}{d r}+k^{2}-\frac{l(l+1)}{r^{2}}\right) f_{l m}(r)=0
\end{gathered}
$$

remark: The sum ores $l$ actually starts at $l=1$.

Solution:

$$
\begin{aligned}
& f_{e m}(r)=a_{e m} j_{e}\left(k_{r}\right)+b_{e m} n_{e}(k r) \\
& j_{l}(x) \equiv \sqrt{\frac{\pi}{2 x}} J_{e+\frac{1}{2}}(x) \\
& n_{l}(x) \equiv \sqrt{\frac{\pi}{2 x}} N_{l+\frac{1}{2}}(x) \\
& h_{l}^{(1)}(x) \equiv \sqrt{\frac{\pi}{2 x}}\left[J_{l+\frac{1}{2}}(x)+i N_{l+\frac{1}{2}}(x)\right] \\
& h_{l}^{(2)}(x) \equiv \sqrt{\frac{\pi}{2 x}}\left[J_{l+\frac{1}{2}}(x)-i N_{l+\frac{1}{2}}(x)\right]
\end{aligned}
$$

Explicitly

$$
\begin{aligned}
& f_{e}(x)=(-x)^{e}\left(\frac{1}{x} \frac{d}{d x}\right)^{e}\left(\frac{\sin x}{x}\right) \\
& n_{l}(x)=-(-x)^{e}\left(\frac{1}{x} \frac{d}{d x}\right)^{e}\left(\frac{\cos x}{x}\right)
\end{aligned}
$$

For example:

$$
\begin{aligned}
& \partial_{0}(x)=\frac{\sin x}{x} \\
& f_{1}(x)=\frac{\sin x}{x^{2}}-\frac{\cos x}{x}
\end{aligned}
$$

Large argument limit $\quad x \rightarrow \infty$

$$
\begin{aligned}
& j_{e}^{\prime}(x) \rightarrow \frac{1}{x} \sin \left(x-\frac{l \pi}{2}\right) \\
& n_{l}(x) \rightarrow \frac{-1}{x} \cos \left(x-\frac{l \pi}{2}\right) \\
& h_{e}^{(1)}(x) \longrightarrow(-i)^{l+1} \frac{e^{i x}}{x}
\end{aligned}
$$

$x=k r$
outgoing spherical waves $(r \rightarrow \infty)$

$$
\sim \frac{e^{i k r}}{r}
$$

Conclusion

$$
\begin{aligned}
& \psi^{E}(\vec{x})=\sum_{e m} \psi_{e m}^{E} h_{e}^{(1)}(k r) Y_{e m}(\theta, \phi) \\
& \psi^{m}(\vec{x})=\sum_{e m} \psi_{e m}^{m} h_{e}^{(1)}(k r) y_{e m}(\theta, \phi)
\end{aligned}
$$

sum starts at $l=1$

$$
\begin{aligned}
\vec{B}(\vec{x})= & \sum_{e m} \psi_{e m}^{E} h_{e}^{(1)}(k r) \vec{L} Y_{e m}(\Omega) \\
& +\sum_{e m} \psi_{e m}^{m}\left(\frac{-i}{k}\right) \vec{\nabla} \times \vec{L} A_{e}^{(1)}(k r) Y_{e m}(\Omega) \\
\vec{E}(\vec{x})= & \sum_{e m} \psi_{e m}^{E} \frac{i}{k} \vec{\nabla} \times \vec{L} h_{e}^{(1)}(k r) Y_{e m}(\Omega) \\
& +\sum_{e m} \psi_{e m}^{m} h_{e}^{(1)}(k r) \vec{L} Y_{e m}(\Omega)
\end{aligned}
$$

In the rackation zone, $r \rightarrow \infty$

$$
h_{e}^{(1)}(k r)=(-i)^{l+1} \frac{e^{i k r}}{k r}
$$

Electric $(l, m)$ multipole stast at $l=1$

$$
\begin{aligned}
& \vec{B}_{e m}^{(E)}=(-i)^{e+1} \frac{e^{i k r}}{k r} \vec{L}_{l e m}(\theta, \phi) \\
& \vec{E}_{l m}^{(E)}=\frac{i}{k} \vec{\nabla} \times B_{e m}^{(E)}
\end{aligned}
$$

Magnetic $(l, m)$ multipole $s t a s t$ at $l=1$

$$
\begin{aligned}
& \vec{E}_{e m}^{(m)}=(-i)^{l+1} \frac{e^{e k r}}{k r} \vec{L} Y_{e m}(\theta, \phi) \\
& \vec{B}_{l m}^{(m)}=-\frac{i}{k} \vec{\nabla} \times \vec{E}_{e m}^{(m)}
\end{aligned}
$$

Vector spherical harmonics

$$
\begin{aligned}
& \vec{X}_{l m}(\theta, \phi) \equiv \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{l m}(\theta, \phi) \\
& \ell=1,2, \ldots \\
& m=-l,-l+1, \ldots, \ell-1, l \\
& \int d \Omega \vec{X}_{l^{\prime} m^{\prime}}^{*}(\theta, \phi) \cdot \vec{X}_{l m}(\theta, \phi)=\delta_{l \ell^{\prime}} \delta_{m m \prime}
\end{aligned}
$$

Note $\vec{\nabla} \cdot \vec{X}_{\text {em }}(\theta, \phi)=0$ since $\vec{D} \cdot \vec{L}=0$
second vector spherical harmonic $i \vec{D} \times \vec{L} Y_{\text {em }}(\theta, \phi)$

Last time, we identified the electric multipoles

$$
(l, m) \quad l=1,2,3, \ldots \text { and } m=-l,-l+1, \ldots, l-1, l
$$

with magnetic and electric fields given by

$$
\begin{aligned}
& \vec{B}(\vec{x})=\psi_{e m}^{E} \vec{B}_{e m}^{(E)}(\vec{x}) \\
& \vec{E}(\vec{x})=\psi_{e m}^{E} \vec{E}_{e m}^{(E)}(\vec{x})
\end{aligned}
$$

$$
\begin{aligned}
& \vec{B}(\vec{x}, t)=\vec{B}(\vec{x}) e^{-i \omega t} \\
& \vec{E}(\vec{x}, t)=\vec{E}(\vec{x}) e^{-i \omega t}
\end{aligned}
$$

where

$$
\begin{aligned}
& \vec{B}_{e m}^{(E)}(\vec{x})=(-i)^{e+1} \frac{e^{i k r}}{k r} \vec{L} Y_{e m}(\Omega) \\
& \vec{E}_{\text {em }}^{(E)}(\vec{x})=\frac{i}{k} \vec{\nabla} \times \vec{B}_{e m}^{(\epsilon)} \\
& \text { diction zone }(k r \gg 1) .
\end{aligned}
$$

in the radiation zone $(k r \gg 1)$.
The vector spherical harmonics ane defined by

$$
\vec{x}_{e m}(\Omega) \equiv \frac{1}{\sqrt{e(l+1)}} \vec{L} Y_{e m}(\Omega)
$$

$$
\begin{aligned}
\vec{L} & =-c \vec{x} \times \vec{D} \\
& =-c r \hat{n} \times \vec{D}
\end{aligned}
$$

To make contact with Jackson's notation

$$
a_{E}(l, m) \equiv \sqrt{l(l+1)} \psi_{\mathrm{em}}^{E}
$$

Likewise, the magnetic multipoles fields are

$$
\begin{aligned}
& \vec{E}(\vec{x})=\psi_{e m}^{m} \vec{E}_{e m}^{(m)}(\vec{x}) \\
& \vec{B}(\vec{x})=\psi_{e m}^{m} \vec{B}^{(m)}(\vec{x})
\end{aligned}
$$

where

$$
\vec{E}_{e m}^{(m)}(\vec{x})=(-i)^{e+1} \frac{e^{i k r}}{k r} \vec{L} Y_{e m}(\Omega)
$$

$$
\vec{B}_{e m}^{(m)}(\vec{x})=-\frac{i}{k} \vec{\nabla} \times \vec{E}_{e m}^{(m)}
$$

in the radiation zone ( $k r \gg 1$ ).
Making contact with Jackson's notation,

$$
a_{m}(\ell, m) \equiv \sqrt{l(\ell+1)} \psi_{e m}^{m}
$$

Rachated power

$$
\begin{aligned}
P & =\oint \vec{S} \cdot d \vec{a} \\
\frac{d P}{d \Omega} & =\frac{c}{8 \pi} \operatorname{Re}\left(\vec{E} \times \vec{B}^{*}\right) \cdot \hat{n} r^{2}
\end{aligned}
$$

(i) electric multipole radiation $(l, m)$

Need

$$
\begin{aligned}
& \frac{\lambda}{k} \vec{\nabla} \times\left[\frac{e^{i k r}}{r} \vec{L} Y_{e m}(\Omega)\right] \\
= & \frac{\lambda}{k} \vec{\nabla}\left(\frac{e^{i k r}}{r}\right) \times \vec{L} Y_{l m}(\Omega)+\frac{e^{i k r}}{k r} \vec{i} \times \vec{L} Y_{l m}(\Omega)
\end{aligned}
$$

useful operator identity

$$
i \vec{\nabla} \times \vec{L}=\vec{x} \vec{\nabla}^{2}-\vec{\nabla}\left(1+r \frac{\partial}{\partial r}\right)
$$

$$
\begin{gathered}
\vec{\nabla}\left(\frac{e^{i k r}}{r}\right)=\hat{n} \frac{\partial}{\partial r}\left(\frac{e^{i k r}}{r}\right)=i k \frac{e^{i k r}}{r} n+o\left(\frac{1}{r^{2}}\right) \\
\frac{\partial}{\partial r} y_{e m}(\Omega)=0
\end{gathered}
$$

Thus,

$$
\begin{aligned}
& \frac{i}{k} \vec{\nabla} \times\left[\frac{e^{i k r}}{r} \vec{L} Y_{e m}(\Omega)\right] \\
& =-\frac{e^{i k r}}{k r}\left[k \hat{n} \times \vec{L} Y_{e m}(\Omega)-\left(\vec{x} \vec{\nabla}^{2}-\vec{\nabla}\right) Y_{e m}(\Omega)\right] \\
& \left.=-\frac{1}{r^{2}}\right) \\
& =\frac{e^{i k r}}{r} \hat{n} \times \vec{L} Y_{e m}(\Omega)+O\left(\frac{1}{r^{2}}\right)
\end{aligned}
$$

[e.g. $\left.\vec{\nabla}=\hat{n} \frac{\partial}{\partial r}+\frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta}+\frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi}\right]$
That is,

$$
\begin{aligned}
& \vec{E}_{l m}^{E}=-\hat{n} \times \vec{B}_{l m}^{\epsilon}+o\left(\frac{1}{r^{2}}\right) \\
& \vec{E} \times \vec{B}^{*}=-(\hat{n} \times \vec{B}) \times \vec{B}^{*}=\hat{n}|\vec{B}|^{2}-\vec{B}\left(\hat{n} \cdot \vec{B}^{*}\right) \\
&=\hat{n}|\vec{B}|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \vec{S} \cdot n \cdot r^{2} d \Omega=\frac{c}{8 \pi k^{2}}\left|\psi_{e m}^{E}\right|^{2}\left|\vec{L} Y_{e m}(\Omega)\right|^{2} d \Omega \\
& \frac{d P_{E e_{m}}}{d \Omega}=\frac{c}{8 \pi k^{2}}\left|\psi_{e m}^{E}\right|^{2} l(l+1)\left|\vec{X}_{e m}(\Omega)\right|^{2}
\end{aligned}
$$

Integrating over angles

$$
\begin{aligned}
& \int d \Omega \vec{X}_{e^{\prime} m^{\prime}}^{*}(\Omega) \cdot \vec{X}_{l m}(\Omega)=\delta_{l \ell^{\prime}} \delta_{m m^{\prime}} \\
\Rightarrow & \int d \Omega\left|\vec{X}_{e m}(\Omega)\right|^{2}=1 \\
& P_{E l m}=\frac{c l(l+1)}{8 \pi k^{2}}\left|\psi_{e m}^{E}\right|^{2}
\end{aligned}
$$

(ii) magnetic multipole radiation

$$
\begin{aligned}
& \vec{B} \rightarrow \vec{E}, \vec{E} \rightarrow-\vec{B} \\
& \left.\frac{d P_{\text {mem }}}{d \Omega}=\frac{c}{8 \pi R^{a}}\left|\psi_{l m}^{m}\right|^{2} l(\ell+1)\left|\vec{X}_{\text {em }}(\Omega)\right|^{2} \right\rvert\,
\end{aligned}
$$

